Taylor Series Example: Efficient choice of "a"

Suppose you want to approximate $\sin\left(\frac{8\pi}{5}\right)$ to within 0.00001.

You can use the Maclaurin Series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+1}, \qquad (-\infty, \infty)$$

with x= $\frac{8\pi}{5}$, so $\sin\left(\frac{8\pi}{5}\right) = \left(\frac{8\pi}{5}\right) - \frac{\left(\frac{8\pi}{5}\right)^3}{3!} + \frac{\left(\frac{8\pi}{5}\right)^5}{5!} - \frac{\left(\frac{8\pi}{5}\right)^7}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{8\pi}{5}\right)^{2n+1}, \qquad (-\infty, \infty)$

However, to get the desired accuracy, you would need the 21st degree polynomial since the first term of this alternating series which is less than 0.00001 is $\frac{\left(\frac{8\pi}{5}\right)^{23}}{23!} \approx 0.0000005$ Thus you would have to compute:

$$\sin\left(\frac{8\pi}{5}\right) \approx \left(\frac{8\pi}{5}\right) - \frac{\left(\frac{8\pi}{5}\right)^3}{3!} + \frac{\left(\frac{8\pi}{5}\right)^5}{5!} - \frac{\left(\frac{8\pi}{5}\right)^7}{7!} + \ldots + \frac{\left(\frac{8\pi}{5}\right)^{21}}{21!}$$

That is a lot of work!! A more efficient way to approximate $\sin\left(\frac{8\pi}{5}\right)$ is to chose a value of "a" in the Taylor series formula $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ that is near $\frac{8\pi}{5}$ for which the sine is easily computed. Notice $\frac{3\pi}{2} = \frac{15\pi}{10}$ which is near $\frac{16\pi}{10} = \frac{8\pi}{5}$, and $\sin\left(\frac{3\pi}{2}\right)$ is easily computed, so find the Taylor Series for sinx with a= $\frac{3\pi}{2}$.

$$\sin x = -1 + \frac{\left(x - \frac{3\pi}{2}\right)^2}{2!} - \frac{\left(x - \frac{3\pi}{2}\right)^4}{4!} + \frac{\left(x - \frac{3\pi}{2}\right)^6}{6!} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{(2n)!} \left(x - \frac{3\pi}{2}\right)^{2n}, \qquad (-\infty, \infty)$$

so

$$\sin\left(\frac{8\pi}{5}\right) = -1 + \frac{\left(\frac{8\pi}{5} - \frac{3\pi}{2}\right)^2}{2!} - \frac{\left(\frac{8\pi}{5} - \frac{3\pi}{2}\right)^4}{4!} + \frac{\left(\frac{8\pi}{5} - \frac{3\pi}{2}\right)^6}{6!} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{(2n)!} \left(\frac{8\pi}{5} - \frac{3\pi}{2}\right)^{2n}, \qquad (-\infty, \infty)$$

$$= -1 + \frac{\left(\frac{\pi}{10}\right)^2}{2!} - \frac{\left(\frac{\pi}{10}\right)^4}{4!} + \frac{\left(\frac{\pi}{10}\right)^6}{6!} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{(2n)!} \left(\frac{\pi}{10}\right)^{2n}, \qquad (-\infty, \infty)$$

In this case the first term that is less than 0.00001 is $\frac{\left(\frac{\pi}{10}\right)^6}{6!} \approx 0.000001$. Thus $\sin\left(\frac{8\pi}{5}\right)$ can be approximated by simply using the 4th degree Taylor polynomial

$$\sin\left(\frac{8\pi}{5}\right) = -1 + \frac{\left(\frac{\pi}{10}\right)^2}{2!} - \frac{\left(\frac{\pi}{10}\right)^4}{4!} \approx -0.9510578$$